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Scale Transformations on the Noncommutative Plane and the Seiberg-Witten Map

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ABSTRACT

We write down three kinds of scale transformations `i-iii)` on the noncommutative plane. `i)` is the analogue of standard dilations on the plane, `ii)` is a re-scaling of the noncommutative parameter θ, and `iii)` is a combination of the previous two, whereby the defining relations for the noncommutative plane are preserved. The action of the three transformations is defined on gauge fields evaluated at fixed coordinates and θ. The transformations are obtained only up to terms which transform covariantly under gauge transformations. We give possible constraints on these terms. We show how the transformations `i)` and `ii)` depend on the choice of star product, and show the relation of `ii)` to Seiberg-Witten transformations. Because `iii)` preserves the fundamental commutation relations it is a symmetry of the algebra. One has the possibility of implementing it as a symmetry of the dynamics, as well, in noncommutative field theories where θ is not fixed.

1 Introduction

Noncommutative field theory is incompatible with conformal field theory. Moreover conformal symmetry is violated in noncommutative theories, at least in their usual formulation, by the presence of dimensionfull parameters. Here we shall be concerned, in particular, with effects due to dilations. For the example of the noncommutative plane, standard scale transformations of its coordinates $\mathbf{x}_i,\ i = 1, 2$, do not preserve the defining commutation relations

$$[\mathbf{x}_i, \mathbf{x}_j] - i\theta\epsilon_{ij} = 0 \quad , \tag{1.1}$$

where θ is the dimensionfull parameter, known as the noncomutativity parameter, and it characterizes the noncommutative plane. A re-scaling of θ, corresponding to a mapping from one noncommutative plane to another, also does not preserve (1.1). Such a re-scaling is generally associated with the Seiberg-Witten map[1]. On the other hand, we can preserve (1.1) with a *simultaneous* dilation of the coordinates and a re-scaling of θ. Such transformations then define a symmetry of the algebra. Moreover, it may also be possibility to implement such transformations as a symmetry of the dynamics. Since the transformations involve a change in θ, as well as $\mathbf{x}_i$, they act on an ensemble of noncommutative planes, rather than a single noncommutative plane. This approach may allow one to recover an analogue of conformal symmetry within the context of noncommutative field theory.*

Concerning the Seiberg-Witten map, gauge fields are introduced on the noncommutative plane. Their algebra can be realized as functions (or symbols) on the commutative plane by working with some associative star product. It was shown in [1] that the symbols $\mathcal{A}_i$ associated with the noncommutative potentials could be expressed in terms of commutative potentials $\mathcal{A}^c_i$, along with their derivatives, and the noncommutative gauge parameter λ could be expressed in terms of the commutative one λ^c and $\mathcal{A}^c_i$, along with their derivatives. In Abelian gauge theory the commutative potentials gauge transform as $\mathcal{A}^c_i \to \mathcal{A}^c_i + \partial_i\lambda^c$, which then induces a transformation in the noncommutative potentials $\mathcal{A}_i(\mathcal{A}^c) \to \mathcal{A}_i(\mathcal{A}^c + \partial\lambda^c)$. In the Seiberg-Witten equations the latter is identified with a noncommutative gauge transformation of the potentials $\mathcal{A}_i$. At first order in the noncomutativity parameter θ we don't need to specify the star product. So at first order an infinitesimal gauge variation is given by

$$\begin{aligned} \delta^g_\lambda\, \mathcal{A}_i(\mathcal{A}^c) &= \mathcal{A}_i(\mathcal{A}^c + \partial\lambda^c) \;-\; \mathcal{A}_i(\mathcal{A}^c) \\ &= \partial_i\lambda(\lambda^c, \mathcal{A}^c) \;+\; \{\lambda(\lambda^c, \mathcal{A}^c),\ \mathcal{A}_i(\mathcal{A}^c)\}\ , \end{aligned} \tag{1.2}$$

where $\{\ ,\ \}$ denotes the Poisson bracket. For any two functions $\mathcal{F}$ and $\mathcal{G}$ on the commutative plane, it is given by

$$\{\mathcal{F}, \mathcal{G}\} = \theta\ \epsilon_{ij}\partial_i\mathcal{F}\partial_j\mathcal{G}\ , \tag{1.3}$$

*For another approach see [2].

where ∂_i are derivatives with respect to the coordinates on the plane. The first order solutions for the maps $\mathcal{A}_i(\mathcal{A}^c)$ and $\lambda(\lambda^c, \mathcal{A}^c)$ are

$$\begin{aligned}
\mathcal{A}_i(\mathcal{A}^c) &= \mathcal{A}_i^c - \frac{\theta}{2}\epsilon_{jk}\, \mathcal{A}_j^c\, (\mathcal{F}_{ik}^c - \partial_k \mathcal{A}_i^c) + \frac{1}{2}\mathcal{H}^{(2)}_{\mathcal{A}_i^c} \\
\lambda(\lambda^c, \mathcal{A}^c) &= \lambda^c + \frac{\theta}{2}\,\epsilon_{ij}\, \mathcal{A}_i^c\, \partial_j \lambda^c \ ,
\end{aligned} \tag{1.4}$$

$\mathcal{F}_{ij}^c$ being the commutative curvature. We call $\mathcal{H}^{(2)}_{\mathcal{A}_i^c}$ a homogenous term. It is only required to satisfy

$$\mathcal{H}^{(2)}_{\mathcal{A}_i^c+\partial_i\lambda^c} - \mathcal{H}^{(2)}_{\mathcal{A}_i^c} = \{\lambda^c, \mathcal{H}^{(2)}_{\mathcal{A}_i^c}\} \ , \tag{1.5}$$

which corresponds to the first order noncommutative gauge transformation of a covariant field. Analogous homogeneous terms appear in the Seiberg-Witten map of matter fields. In general the homogeneous terms are undetermined, and such ambiguities in the construction of Seiberg-Witten map are well known[3],[4],[5],[6],[7], although they are often ignored in the literature.

Arbitrary homogeneous terms also result upon making dilations of the coordinates $\mathbf{x}_i$ of the noncommutative plane, as well simultaneous dilations and scale transformations in θ. Here we find a number of relations connecting the various homogeneous terms. We can get additional constraints on the homogenous terms if we demand that the gauge fields carry a faithful representation of the two independent scale transformations, dilations of $\mathbf{x}_i$ and re-scalings of θ. The constraints allow for nontrivial solutions, although the constraints are insufficient in removing all the ambiguities in the homogeneous terms. More constraints may result from the presence of other symmetries, and they may help fix further degrees of freedom in the homogeneous terms. Upon generalizing to higher orders in θ, it becomes necessary to specify the choice of star product, as the answer depends on this choice. We show the explicit dependence of the transformations on the choice of star product.

In section 2 we write down the different types of scale transformations on the noncommutative plane. A fundamental issue is the construction of operators generating the various transformations. Concerning the generator D of simultaneous dilations and re-scalings of θ, we obtain the most general operator that a) satisfies the Leibniz rule when acting on the product of two functions on the noncommutative plane $\times\ \mathbb{R}^1$, $\mathbb{R}^1$ parametrized by θ, and b) annihilates the left hand side of (1.1). We can then say that D is a generator of a symmetry of the algebra. In section 3 we show how gauge fields transform under these scale transformations. Our approach closely follows that of Grimstrup, Jonsson and Thorlacius [7], in that we require the commutator of gauge transformations with scale transformations to close to gauge transformations. This requirement then insures that all gauge invariant quantities remain gauge invariant under scale transformations. As an example, we write down a one parameter family of Seiberg-Witten maps which interpolate between different star products. We then write down constraints on the homogeneous terms and giving some explicit solutions. We conclude in section 4 with some preliminary remarks on the possibility of implementing simultaneous dilations and re-scalings in θ as a symmetry of the dynamics, as well as the algebra. This

symmetry is not a deformation of the standard dilation symmetry on the plane, but rather a new symmetry on the noncommutative plane $\times \mathbb{R}$.

2 Three Scale Transformations

For simplicity we begin with the noncommutative plane at first order in 2.1, and then discuss the fully noncommutative case in 2.2.

2.1 Noncommutative Plane at First Order

The family of noncommutative planes at first order can be defined as $\mathbb{R}^3$ modded out by an equivalence relation. Let $\mathbb{R}^3$ be parametrized by $(x_1,\ x_2,\ \theta)$. Then the equivalence relation is

$$\{x_i, x_j\} - \epsilon_{ij}\,\theta = 0\;, \tag{2.1}$$

and it is the commutative limit of (1.1). $\{\ ,\ \}$ once again denotes the Poisson bracket defined in (1.3), which is degenerate on $\mathbb{R}^3$.

We consider three separate scale transformations on $\mathbb{R}^3$ parametrized in each case by a real number ρ:

$$\begin{aligned} \texttt{i)}\quad (x,\theta) &\rightarrow (\rho^{-1}x,\theta) \\ \texttt{ii)}\quad (x,\theta) &\rightarrow (x,\rho^{-2}\theta) \\ \texttt{iii)}\quad (x,\theta) &\rightarrow (\rho^{-1}x,\rho^{-2}\theta) \end{aligned} \tag{2.2}$$

`i)` is a standard dilation of the coordinates, `ii)` scales θ and thus maps to new noncommutative plane, while `iii)` scales both x_i and θ in such a way that it leads to an automorphism of the algebra defined by (2.1). `i)` and `ii)` are independent transformations, while `iii)` is a combination of `i)` and `ii)`.

Next introduce representations of these transformations on fields ϕ on $\mathbb{R}^3$:

$$\begin{aligned} \texttt{i)}\quad \phi(x,\theta) &\rightarrow {}^{\rho_1}\phi(x,\theta) = e^{\chi^1_{\rho,\mathcal{A}}}\,\phi(\rho x,\theta) \\ \texttt{ii)}\quad \phi(x,\theta) &\rightarrow {}^{\rho_2}\phi(x,\theta) = e^{\chi^2_{\rho,\mathcal{A}}}\,\phi(x,\rho^2\theta) \\ \texttt{iii)}\quad \phi(x,\theta) &\rightarrow {}^{\rho_3}\phi(x,\theta) = e^{\chi^3_{\rho,\mathcal{A}}}\,\phi(\rho x,\rho^2\theta)\;, \end{aligned} \tag{2.3}$$

where $\chi^a_{\rho,\mathcal{A}}\ ,\quad a = 1,2,3$ are ρ dependent operators acting on the space of fields. The $\mathcal{A}$ subscript indicates that representations are, in general, not diagonal, and that transformations on some field ϕ may involve additional fields $\mathcal{A}_i$. `ii)` is related to a Seiberg-Witten map. In

the latter, however, the transformed field is standardly evaluated at the transformed value of θ, i.e.

$$\texttt{SW)} \quad \phi(x,\theta) \to {}^{\rho_2}\phi(x,\rho^{-2}\theta) \tag{2.4}$$

Setting $\rho = 1-\epsilon$, ϵ being infinitesimal, gives the infinitesimal version,

$$\phi(x,\theta) \to \phi(x,\theta) + \delta^{s_a}_\epsilon \phi(x,\theta)\ , \tag{2.5}$$

of transformations `i)-iii)`, where

$$\begin{aligned}
\texttt{i)} \quad \delta^{s_1}_\epsilon &= \epsilon\,(\chi^1_{\mathcal{A}} - x_i\partial_i)\ , \qquad \partial_i = \frac{\partial}{\partial x_i} \\
\texttt{ii)} \quad \delta^{s_2}_\epsilon &= \epsilon\,(\chi^2_{\mathcal{A}} - 2\theta\partial_\theta)\ , \qquad \partial_\theta = \frac{\partial}{\partial \theta} \\
\texttt{iii)} \quad \delta^{s_3}_\epsilon &= \epsilon\,(\chi^3_{\mathcal{A}} - D)\ , \qquad D = x_i\partial_i + 2\theta\partial_\theta\ ,
\end{aligned} \tag{2.6}$$

with $\chi^a_{\mathcal{A}} = \lim_{\epsilon\to 0}\frac{\chi^a_{\rho,\mathcal{A}}}{\epsilon}$. The infinitesimal variations $\delta^{s_a}_\epsilon$ are related by $\delta^{s_3}_\epsilon = \delta^{s_1}_\epsilon + \delta^{s_2}_\epsilon$, leading to the constraint on $\chi^a_{\mathcal{A}}$

$$\chi^3_{\mathcal{A}} = \chi^1_{\mathcal{A}} + \chi^2_{\mathcal{A}} \tag{2.7}$$

When acting on the Poisson bracket of two functions $\mathcal{F}$ and $\mathcal{G}$ on $\mathbb{R}^3$, D satisfies the Leibniz rule

$$D\{\mathcal{F},\mathcal{G}\}(x,\theta) = \{D\mathcal{F},\mathcal{G}\}(x,\theta) + \{\mathcal{F},D\mathcal{G}\}(x,\theta) \tag{2.8}$$

Variations $\delta^{s_a}_\epsilon$ also satisfy the Leibniz rule as they are evaluated at fixed coordinates in $\mathbb{R}^3$. Using (2.8), D is seen to annihilate the left hand side of (2.1), and so transformations `iii)` leave invariant the equivalence relation. On the other hand, $x_i\partial_i$ and $2\theta\partial_\theta$ do not satisfy the Leibniz rule, but rather:

$$x_i\partial_i\{\mathcal{F},\mathcal{G}\}(x,\theta) = \{x_i\partial_i\mathcal{F},\mathcal{G}\}(x,\theta) + \{\mathcal{F},x_i\partial_i\mathcal{G}\}(x,\theta) - 2\{\mathcal{F},\mathcal{G}\}(x,\theta) \tag{2.9}$$

$$2\theta\partial_\theta\{\mathcal{F},\mathcal{G}\}(x,\theta) = \{2\theta\partial_\theta\mathcal{F},\mathcal{G}\}(x,\theta) + \{\mathcal{F},2\theta\partial_\theta\mathcal{G}\}(x,\theta) + 2\{\mathcal{F},\mathcal{G}\}(x,\theta) \tag{2.10}$$

Note that with these modified product rules, $x_i\partial_i$ and $2\theta\partial_\theta$ also annihilate the left hand side of (2.1), and hence leave invariant the equivalence relation.

The infinitesimal version of the Seiberg-Witten map (2.4) is

$$\texttt{SW)} \quad \phi(x,\theta) \to \phi(x,\theta) + \delta^{SW}\phi(x,\theta)\ , \qquad \delta^{SW} = \delta^{s_2}_\epsilon + 2\epsilon\theta\partial_\theta\ , \tag{2.11}$$

and so this variation is given solely by $\chi^2_{\mathcal{A}}$,

$$\delta^{SW} = \frac{\delta\theta}{2\theta}\,\chi^2_{\mathcal{A}}\ , \tag{2.12}$$

where $\delta\theta = 2\epsilon\theta$. Acting on the Poisson bracket it then does not satisfy the Leibniz rule, but rather

$$\delta^{SW}\{\mathcal{F},\mathcal{G}\} = \{\delta^{SW}\mathcal{F},\mathcal{G}\} + \{\mathcal{F},\delta^{SW}\mathcal{G}\} + \frac{\delta\theta}{\theta}\{\mathcal{F},\mathcal{G}\} \tag{2.13}$$

In the next subsection we show how this result generalizes in the full noncommutative theory. In that case, the Seiberg-Witten variation δ^{SW} acting on a product of functions violates the Leibniz rule. This is since the variation δ^{SW} compares functions at different values of θ.

2.2 Noncommutative Plane to All Orders

To go to all orders we replace x_i by operators $\mathbf{x}_i$, $i = 1, 2$. The latter satisfy (1.1) and generate the associative algebra corresponding to the two-dimensional Moyal (or noncommutative) plane. The Moyal plane is characterized by θ, which remains a c-number. We consider the analogue of the three scale transformations (2.2). `i`) is now a dilation of operators $\mathbf{x}_i$, `ii`) maps between different Moyal planes, and `iii`) scales both $\mathbf{x}_i$ and θ in such a way that it leads to an automorphism of the algebra (1.1).

Concerning `i`), the analogue of the dilation generator $x_i\partial_i$ is ambiguous. $\frac{1}{2}[\mathbf{x}_i, \nabla_i \quad]_+$ was suggested in [7], where $[\ ,\ ]_+$ denotes the anticommutator and ∇_i is the inner derivative on the noncommuting plane. Acting on some function F the latter is given by

$$\nabla_i F = \frac{i}{\theta}\epsilon_{ij}[\mathbf{x}_j, F] \tag{2.14}$$

For a more general noncommutative dilation generator, we add $-2\theta\ \tau$ to $\frac{1}{2}[\mathbf{x}_i, \nabla_i \quad]_+$, where τ is a linear operator acting on the space of functions on the noncommuting plane. Below we will obtain several constraints on τ.

Concerning `ii`) we need to define an analogue of the derivative ∂_θ. We call it ∇_θ. Unlike ∇_i, it is not an inner derivative. We instead define ∇_θ such that it commutes with ∇_i and it is the ordinary derivative on a c-number valued function f of θ, i.e. $\nabla_\theta f(\theta) = \partial_\theta f(\theta)$. We shall also require that it satisfies a product rule such that ∇_θ annihilates the left hand side of (1.1), and it is consistent with the associativity of the algebra, i.e. $\nabla_\theta((FG)H) = \nabla_\theta(F(GH))$. This product rule is not the Leibniz rule.

With regard to `iii`) we will need to construct the noncommutative analogue of the derivative operator D, which for convenience we also call D. We define it, as in the case of first order noncommutativity, to be the sum of the generators for `i`) and `ii`). Thus acting on function F on the noncommutative plane $\times\ \mathbb{R}^1$ ($\mathbb{R}^1$ being parametrized by θ),

$$DF = \frac{1}{2}[\mathbf{x}_i, \nabla_i F]_+ + 2\theta(\nabla_\theta F - \tau(F)\ ) \tag{2.15}$$

In order to recover the first order result, we need that τ acting on fields vanishes at lowest order in θ. For the fully noncommutative D we shall require that a), unlike ∇_θ, it satisfies the Leibniz rule when acting on the product of two functions F and G on the noncommutative plane $\times\ \mathbb{R}^1$

$$D(FG) = (DF)G + F(DG) \tag{2.16}$$

and b) it annihilates the left hand side of (1.1) when evaluated on the noncommutative plane. For this we need that

$$\tau([\mathbf{x}_i,\mathbf{x}_j]) = [\tau(\mathbf{x}_i),\mathbf{x}_j] + [\mathbf{x}_i,\tau(\mathbf{x}_j)] \tag{2.17}$$

Later in this section we find it convenient to impose stronger conditions on τ:

$$\tau(\mathtt{x}_i) \;\propto\; \text{central element}\;, \qquad \tau(\text{central element}) = 0 \tag{2.18}$$

From a) D is consistent with the associativity of the algebra, and also agrees with (2.8) at first order. From b) D generates a symmetry of the algebra.

Next define fields Φ belonging to a bimodule on the noncommutative plane $\times\, \mathbb{R}^1$, with an associative product. For the analogue of the infinitesimal variations (2.5) and (2.6) we take

$$\Phi(\mathbf{x},\theta) \to \Phi(\mathbf{x},\theta) + \delta^{s_a}_\epsilon \Phi(\mathbf{x},\theta)\;, \tag{2.19}$$

$$\begin{aligned}
\texttt{i)}\quad \delta^{s_1}_\epsilon\Phi &= \epsilon\,(\chi^1_A\Phi - \frac{1}{2}[\mathbf{x}_i,\nabla_i\Phi]_+ + 2\theta\,\tau(\Phi)\,)\;,\\
\texttt{ii)}\quad \delta^{s_2}_\epsilon\Phi &= \epsilon\,(\chi^2_A\Phi - 2\theta\nabla_\theta\Phi)\;,\\
\texttt{iii)}\quad \delta^{s_3}_\epsilon\Phi &= \epsilon\,(\chi^3_A\Phi - D\Phi)
\end{aligned} \tag{2.20}$$

As in the previous subsection we require $\delta^{s_3}_\epsilon = \delta^{s_1}_\epsilon + \delta^{s_2}_\epsilon$ and hence we have the analogue of (2.7).

We saw in (2.9) that ∂_θ does not satisfy the Leibniz rule when acting on the Poisson bracket. Similarly, ∇_θ satisfies a modified product rule in the fully noncommutative theory. The modified product rule for ∇_θ should be consistent with (2.16), as well as (2.9) at first order. We first obtain it for the case $\tau = 0$ and then generalize to arbitrary τ.

1. $\tau = 0$. To determine the product rule for ∇_θ we first write down the product rule for dilations

   $$\frac{1}{2}[\mathbf{x}_i,\nabla_i(FG)]_+ = \frac{1}{2}[\mathbf{x}_i,\nabla_i F]_+ G + \frac{1}{2}F[\mathbf{x}_i,\nabla_i G]_+ - i\theta\epsilon_{ij}\nabla_i F\nabla_j G \tag{2.21}$$

   We note that with this product rule, dilations annihilate the left hand side of (1.1). For consistency with (2.16) we need the following product rule for ∇_θ:

   $$\nabla_\theta(FG) = (\nabla_\theta F)G + F(\nabla_\theta G) + \frac{i}{2}\epsilon_{ij}\nabla_i F\nabla_j G \tag{2.22}$$

   With this product rule ∇_θ annihilates the left hand side of (1.1). Furthermore, it is easily seen that the product rule (2.22) is consistent with the associativity of the algebra. This also follows from the fact that (2.22) agrees with the Moyal-Weyl star product realization of the operator algebra[8]. For this functions F, G,... on the noncommuting plane $\times\, \mathbb{R}^1$ are replaced by symbols $\mathcal{F}_0$, $\mathcal{G}_0$,... in the Moyal-Weyl star product representation, which

are functions on the commuting plane $\times\,\mathbb{R}^1$. As the star product depends on θ, ∇_θ acts nontrivially on the star product, in the sense that the Leibniz rule is not satisfied. The Moyal-Weyl star, which we denote by $\star_0$, acting on any two symbols is given by

$$\star_0 = \exp\left\{\frac{i\theta}{2}\epsilon_{ij}\overleftarrow{\partial_i}\ \overrightarrow{\partial_j}\right\} \tag{2.23}$$

$\overleftarrow{\partial_i}$ and $\overrightarrow{\partial_j}$ are left and right derivatives on the commuting plane, respectively. Acting with ∇_θ gives

$$\nabla_\theta \star_0 = \frac{i}{2}\,\epsilon_{ij}\overleftarrow{\partial_i}\ \overrightarrow{\partial_j}\ \star_0 \tag{2.24}$$

Then for any two functions $\mathcal{F}_0$ and $\mathcal{G}_0$ on the plane

$$\nabla_\theta\,(\mathcal{F}_0 \star_0 \mathcal{G}_0) = \nabla_\theta\mathcal{F}_0 \star_0 \mathcal{G}_0 + \mathcal{F}_0 \star_0 \nabla_\theta\mathcal{G}_0 + \frac{i}{2}\,\epsilon_{ij}\partial_i\mathcal{F}_0 \star_0 \partial_j\mathcal{G}_0\ , \tag{2.25}$$

which agrees with (2.22).

2. Arbitrary τ. Now in order to recover (2.16), the product rule for ∇_θ should be changed to

$$\nabla_\theta(FG) = (\nabla_\theta F)G + F(\nabla_\theta G) + \frac{i}{2}\epsilon_{ij}\nabla_i F\nabla_j G + \tau(FG) - \tau(F)G - F\tau(G) \tag{2.26}$$

The extra terms are consistent with the associativity of the product, and using (2.17) ∇_θ again annihilates the left hand side of (1.1). With this choice D acting on the product of two fields again satisfies the Leibniz rule. The choice of (2.26) can also be motivated by a star product on the noncommuting plane, only now it is not the Moyal-Weyl star product. Rather, it is a star product which can be obtained by a general Kontsevich map T from the Moyal-Weyl star product. T is a nonsingular operator which maps any pair of symbols $\mathcal{F}_0$ and $\mathcal{G}_0$ in the Moyal-Weyl star product representation to a new pair of symbols $\mathcal{F}$ and $\mathcal{G}$, which realize the noncommutative algebra with respect to the new star product, which we denote by $\star$, with the fundamental property

$$\mathcal{F}\star\mathcal{G} = T(\mathcal{F}_0 \star_0 \mathcal{G}_0)\ , \qquad \mathcal{F} = T(\mathcal{F}_0)\ , \qquad \mathcal{G} = T(\mathcal{G}_0) \tag{2.27}$$

Variations in θ of a symbol $\mathcal{F}$ can be expressed as

$$\delta\mathcal{F} = d\theta\ \nabla_\theta\mathcal{F} = \delta T(\mathcal{F}_0) + T(\delta\mathcal{F}_0) = d\theta\ t(\mathcal{F}) + T(\delta\mathcal{F}_0)\ , \tag{2.28}$$

where $t = \frac{\delta T}{\delta\theta}T^{-1}$. Applying this to the star product of $\mathcal{F}$ and $\mathcal{G}$ and using (2.25) gives

$$\begin{aligned}\delta(\mathcal{F}\star\mathcal{G}) &= d\theta\ t(\mathcal{F}\star\mathcal{G}) + T(\ \delta(\mathcal{F}_0\star_0\mathcal{G}_0)\ ) \qquad (2.29)\\ &= d\theta\ t(\mathcal{F}\star\mathcal{G}) + T\bigg(\ \delta\mathcal{F}_0\star_0\mathcal{G}_0 + \mathcal{F}_0\star_0\delta\mathcal{G}_0 + \frac{i}{2}d\theta\ \epsilon_{ij}\partial_i\mathcal{F}_0\star_0\partial_j\mathcal{G}_0\bigg)\\ &= d\theta\ t(\mathcal{F}\star\mathcal{G}) + T(\delta\mathcal{F}_0)\star\mathcal{G} + \mathcal{F}\star T(\delta\mathcal{G}_0) + \frac{i}{2}d\theta\ \epsilon_{ij}T(\partial_i\mathcal{F}_0)\star T(\partial_j\mathcal{G}_0)\end{aligned}$$

Finally using (2.28) and assuming that T commutes with ∂_i,

$$\delta(\mathcal{F}\star\mathcal{G}) = \delta\mathcal{F}\star\mathcal{G}+\mathcal{F}\star\delta\mathcal{G}+d\theta\left(t(\mathcal{F}\star\mathcal{G})-t(\mathcal{F})\star\mathcal{G}-\mathcal{F}\star t(\mathcal{G})+\frac{i}{2}\epsilon_{ij}\partial_i\mathcal{F}\star\partial_j\mathcal{G}\right)\,, \tag{2.30}$$

and then

$$\nabla_\theta(\mathcal{F}\star\mathcal{G}) = \nabla_\theta\mathcal{F}\star\mathcal{G}+\mathcal{F}\star\nabla_\theta\mathcal{G}+t(\mathcal{F}\star\mathcal{G})-t(\mathcal{F})\star\mathcal{G}-\mathcal{F}\star t(\mathcal{G})+\frac{i}{2}\epsilon_{ij}\partial_i\mathcal{F}\star\partial_j\mathcal{G} \tag{2.31}$$

This result agrees with (2.26) upon interpreting $t(\mathcal{F})$ and $t(\mathcal{G})$ as the symbols of $\tau(F)$ and $\tau(G)$, respectively, with the product of functions realized by the $\star$. The only assumption used in the above was that T commutes with ∂_i, or equivalently

$$[\nabla_i, \tau] = 0 \tag{2.32}$$

This condition implies (2.18). To prove this let $[\nabla_i,\tau]$ act on $\mathtt{x}_j$. If $i\neq j$, then (2.32) implies $\nabla_i\,\tau(\mathtt{x}_j)=0$, or $\tau(\mathtt{x}_j)$ is a function of only $\mathtt{x}_j$. For $i=j$, (2.32) gives

$$\tau\left(\frac{i}{\theta}\epsilon_{ik}[\mathtt{x}_k,\mathtt{x}_i]\right) = \frac{i}{\theta}\epsilon_{ik}[\mathtt{x}_k,\tau(\mathtt{x}_i)]\,, \qquad \text{no sum on } i, \tag{2.33}$$

or $\tau([\mathtt{x}_k,\mathtt{x}_i]) = [\mathtt{x}_k,\tau(\mathtt{x}_i)]$, with $i\neq k$. Then from (2.17), $[\tau(\mathtt{x}_k),\mathtt{x}_i]=0$, and so the only possibility for $\tau(\mathtt{x}_k)$ is given in (2.18). $\tau(\theta)=0$ then follows from (2.17).

At all orders in θ, infinitesimal Seiberg-Witten transformations are given by

$$\mathtt{SW})\quad \Phi\rightarrow\Phi+\delta^{SW}\Phi\,,\qquad \delta^{SW}=\delta^{s_2}_\epsilon+2\epsilon\theta\nabla_\theta \tag{2.34}$$

The variation δ^{SW} is thus determined by χ^2_A, as in (2.12). χ^2_A acting on a product of functions does not respect the Leibniz rule, and hence neither does δ^{SW}. For the case of arbitrary τ,

$$\delta^{SW}(FG) = (\delta^{SW}F)G+F(\delta^{SW}G)+\delta\theta\left(\frac{i}{2}\epsilon_{ij}\nabla_iF\nabla_jG+\tau(FG)-\tau(F)G-F\tau(G)\right)\,, \tag{2.35}$$

which agrees with (2.13) at first order.

We end this section by giving an example of a nonvanishing τ, or actually a one parameter family of operators τ_β, where β is a real parameter. Consider first a one parameter family of Kontsevich maps $T=T_\beta$ given by

$$T_\beta = \exp\left\{\frac{\beta\theta}{4}\partial_i\partial_i\right\} \tag{2.36}$$

The corresponding family of operators is $\tau_\beta = \frac{\beta}{4}\,\nabla_i\nabla_i$. Here $\tau_\beta(\mathtt{x}_i)=0$, and so condition (2.18) is satisfied.[†] Substituting into (2.26) gives a one parameter family of product rules

$$\nabla_\theta(FG) = (\nabla_\theta F)G+F(\nabla_\theta G)+\frac{i}{2}\epsilon_{ij}\nabla_iF\nabla_jG+\frac{\beta}{2}\nabla_iF\nabla_iG \tag{2.37}$$

[†]Concerning the requirement that linear operators τ vanish at lowest order in θ, we show in section 3.3 that this is the case for τ_β acting on fields belonging to nontrivial representations of the gauge group, up to a gauge transformation and covariant terms, the latter of which can be absorbed in the homogeneous terms.

The one parameter family of product rules corresponds to a one parameter family of star products,

$$\star_\beta = \exp\left\{\frac{\theta}{2}\,[i\epsilon_{ij}\overleftarrow{\partial_i}\,\overrightarrow{\partial_j} + \beta\,\overleftarrow{\partial_i}\,\overrightarrow{\partial_i}]\right\} \tag{2.38}$$

The case of $\beta = 1$ corresponds to the Voros star product[9],[10],[11] $\star_1$, which can be written

$$\star_1 = \exp\left\{\theta\,\frac{\overleftarrow{\partial}}{\partial\zeta}\,\frac{\overrightarrow{\partial}}{\partial\bar{\zeta}}\right\}\,, \qquad \zeta = \frac{x_1 + ix_2}{\sqrt{2}}\,, \quad \bar{\zeta} = \frac{x_1 - ix_2}{\sqrt{2}} \tag{2.39}$$

Thus $\star_\beta$ is a one-parameter family of star products connecting the Voros and Moyal Weyl products.

3 Gauge Transformations

Next we introduce $U(1)$ gauge theory in the fully noncommutative theory. χ^a_A can be determined for any given star product, up to homogeneous terms, by demanding that the commutator of transformations (2.20) with gauge transformations is also a gauge transformation.[‡] [7] This requirement then insures that all gauge invariant quantities remain gauge invariant under scale transformations $\mathtt{i}-\mathtt{iii}$).

In 3.1 below we consider the case of fields Φ which are covariant under gauge transformations. This means that infinitesimal gauge transformations parametrized by Λ (an infinitesimal function of the noncommutative plane $\times\,\mathbb{R}^1$) of Φ are given by

$$\Phi \to \Phi^\Lambda = \Phi + \delta^g_\Lambda\Phi\,, \qquad \delta^g_\Lambda\Phi = i[\Lambda,\Phi]\,, \tag{3.1}$$

Potentials A_i and field strength F_{ij} are considered in 3.2. χ^a_A depends in general on potentials, as the notation implies. Under gauge transformations

$$A_i \to A_i^\Lambda = A_i + \delta^g_\Lambda A_i\,, \qquad \delta^g_\Lambda A_i = D_i\Lambda \equiv i[\Lambda, A_i - \epsilon_{ij}\frac{\mathbf{x}_j}{\theta}]\,, \tag{3.2}$$

while F_{ij} transforms covariantly. In 3.3 we give a one parameter family of Seiberg-Witten transformations, while in 3.4 we derive some constraints on the homogeneous contributions to the solutions.

3.1 Covariant fields

In computing the commutator of transformations (2.20) with gauge transformations on a covariant field Φ, we first consider the simpler case of $\tau = 0$, and then the case of arbitrary τ.

[‡] Actually, when obtaining the Seiberg-Witten map it is common to demand the stronger condition that the commutator vanishes .

1. For the case $\tau = 0$ we need the product rule (2.22). The commutators are

$$
\begin{aligned}
[\delta_\epsilon^{s_1}, \delta_\Lambda^g]\,\Phi &= \epsilon\,(-\delta_\Lambda^g \chi_A^1 \Phi + i[\Lambda, \chi_A^1 \Phi] + i[\chi_A^1 \Lambda, \Phi] + \theta \epsilon_{ij}\,[\nabla_i \Lambda, \nabla_j \Phi]_+\,) \\
[\delta_\epsilon^{s_2}, \delta_\Lambda^g]\,\Phi &= \epsilon\,(-\delta_\Lambda^g \chi_A^2 \Phi + i[\Lambda, \chi_A^2 \Phi] + i[\chi_A^2 \Lambda, \Phi] - \theta \epsilon_{ij}\,[\nabla_i \Lambda, \nabla_j \Phi]_+\,) \\
[\delta_\epsilon^{s_3}, \delta_\Lambda^g]\,\Phi &= \epsilon\,(-\delta_\Lambda^g \chi_A^3 \Phi + i[\Lambda, \chi_A^3 \Phi] + i[\chi_A^3 \Lambda, \Phi]\,) \qquad (3.3)
\end{aligned}
$$

These are equal to gauge variations, i.e.

$$[\delta_\epsilon^{s_a}, \delta_\Lambda^g]\,\Phi = \delta^g_{\Lambda_\epsilon^{(a)}}\Phi\,, \qquad (3.4)$$

when

$$
\begin{aligned}
\chi_A^1 \Phi &= \frac{\theta}{2}\epsilon_{ij}[A_i, D_j\Phi + \nabla_j \Phi]_+ + H_\Phi^{(1)} \qquad (3.5) \\
\chi_A^2 \Phi &= -\frac{\theta}{2}\epsilon_{ij}[A_i, D_j\Phi + \nabla_j \Phi]_+ + H_\Phi^{(2)} \qquad (3.6) \\
\chi_A^3 \Phi &= H_\Phi^{(3)}\,, \qquad (3.7)
\end{aligned}
$$

where the covariant derivative is defined in (3.2) and $H_\Phi^{(a)}$ are defined to transform covariantly under gauge transformations, i.e.

$$\delta_\Lambda^g H_\Phi^{(a)} = i[\Lambda, H_\Phi^{(a)}] \qquad (3.8)$$

So possible solutions are $H_\Phi^{(a)}$ proportional to Φ. More generally, $H_\Phi^{(a)}$ can be any polynomial of covariant fields and their covariant derivatives. We call these homogeneous solutions. From (2.7) and (3.5-3.7), the homogenous solutions are related by $H_\Phi^{(3)} = H_\Phi^{(1)} + H_\Phi^{(2)}$. If $H_\Phi^{(3)}$ vanishes, then χ_A^3 annihilates Φ. If $H_\Phi^{(3)}$ is proportional to Φ it is an eigenvector of χ_A^3. The gauge parameters $\Lambda_\epsilon^{(a)}$ corresponding to (3.5), (3.6) and (3.7) are given by

$$
\begin{aligned}
\Lambda_\epsilon^{(1)} &= \epsilon\left(\chi_A^1 \Lambda + \frac{\theta}{2}\epsilon_{ij}[A_i, \nabla_j \Lambda]_+\right) \qquad (3.9) \\
\Lambda_\epsilon^{(2)} &= \epsilon\left(\chi_A^2 \Lambda - \frac{\theta}{2}\epsilon_{ij}[A_i, \nabla_j \Lambda]_+\right) \qquad (3.10) \\
\Lambda_\epsilon^{(3)} &= \epsilon\,\chi_A^3 \Lambda\,, \qquad (3.11)
\end{aligned}
$$

respectively. Had we imposed the stronger condition that scale transformations commute with gauge transformations, then $\Lambda_\epsilon^{(a)} = 0$, and we would obtain unambiguous solutions for $\chi_A^a \Lambda$, and hence $\delta_\epsilon^{s_a}\Lambda$.

2. When $\tau \neq 0$, χ_A^1 gets replaced by $\chi_A^1 + 2\theta\,\tau$, and so (3.5) and (3.9) are generalized to

$$\chi_A^1 \Phi = \frac{\theta}{2}\epsilon_{ij}[A_i, D_j\Phi + \nabla_j \Phi]_+ - 2\theta\tau(\Phi) + H_\Phi^{(1)}\,,$$

$$\Lambda_\epsilon^{(1)} \;=\; \epsilon\left(\chi_A^1\Lambda + 2\theta\tau(\Lambda) + \frac{\theta}{2}\epsilon_{ij}[A_i, \nabla_j\Lambda]_+\right) \tag{3.12}$$

Concerning χ_A^2 , the commutator $[\delta_\epsilon^{s_2}, \delta_\Lambda^g]\,\Phi$ in (3.3) picks up the additional terms

$$2i\epsilon\theta\left(\tau([\Lambda,\Phi]) - [\tau(\Lambda),\Phi] - [\Lambda,\tau(\Phi)]\right),$$

and one can write the answer by replacing χ_A^2 by $\chi_A^2 - 2\theta\,\tau$ in the previous result. As a result (3.6) and (3.10) are generalized to

$$\begin{aligned}
\chi_A^2\Phi &= -\frac{\theta}{2}\epsilon_{ij}[A_i, D_j\Phi + \nabla_j\Phi]_+ + 2\theta\tau(\Phi) + H_\Phi^{(2)}\,, \\
\Lambda_\epsilon^{(2)} &= \epsilon\left(\chi_A^2\Lambda - 2\theta\tau(\Lambda) - \frac{\theta}{2}\epsilon_{ij}[A_i, \nabla_j\Lambda]_+\right)
\end{aligned} \tag{3.13}$$

So up to the homogeneous terms, $\chi_A^1\Phi$ and $\chi_A^2\Phi$ only differ by a sign. In comparing (3.12) and (3.13) it follows that the sum, i.e. $\chi_A^3\Phi$ and the corresponding gauge parameter $\Lambda_\epsilon^{(3)}$, is unaffected by the generalization, and thus still given by (3.7) and (3.11), respectively.

From now on we consider the most general case where τ is not necessarily zero.

From (3.13), using (2.12), we get the general expression for the Seiberg-Witten variation of Φ from χ_A^2. The term $\tau(\Phi)$ depends on the choice of the star product (for example, it vanishes in the case of the Moyal-Weyl star), while $H_\Phi^{(2)}$, represents the homogeneous contributions. At lowest order in θ, the result agrees with the solutions (1.4).

3.2 Potentials and field strength

We next determine the variations $\delta_\epsilon^{s_a}$ of the gauge potentials and field strength. The latter is given by

$$F_{ij} = \nabla_i A_j - \nabla_j A_i - i[A_i, A_j] \tag{3.14}$$

Concerning the potentials,

$$\begin{aligned}
\texttt{i)}\quad \delta_\epsilon^{s_1}A_i &= \epsilon\left(\chi_A^1 A_i - \frac{1}{2}[\mathbf{x}_j, \nabla_j A_i]_+ + 2\theta\,\tau(A_i)\right) \\
\texttt{ii)}\quad \delta_\epsilon^{s_2}A_i &= \epsilon\,(\chi_A^2 A_i - 2\theta\nabla_\theta A_i) \\
\texttt{iii)}\quad \delta_\epsilon^{s_3}A_i &= \epsilon\,(\chi_A^3 A_i - DA_i)
\end{aligned} \tag{3.15}$$

We thus need to determine $\chi_A^a A_i$. For this we can first look at variations $\delta_\epsilon^{s_a}$ of the covariant derivative of Φ

$$\texttt{i)}\quad \delta_\epsilon^{s_1}D_i\Phi \;=\; D_i\delta_\epsilon^{s_1}\Phi + i[\Phi, \delta_\epsilon^{s_1}A_i] = \epsilon\left(\chi_A^1 D_i\Phi - \frac{1}{2}[\mathbf{x}_j, \nabla_j D_i\Phi]_+ + 2\theta\,\tau(D_i\Phi)\right)$$

$$
\begin{aligned}
\texttt{ii)}\quad \delta_\epsilon^{s_2} D_i\Phi &= D_i\delta_\epsilon^{s_2}\Phi + i[\Phi,\delta_\epsilon^{s_2}A_i] = \epsilon\,(\chi_A^2 D_i\Phi - 2\theta\nabla_\theta D_i\Phi) \\
\texttt{iii)}\quad \delta_\epsilon^{s_3} D_i\Phi &= D_i\delta_\epsilon^{s_3}\Phi + i[\Phi,\delta_\epsilon^{s_3}A_i] = \epsilon\,(\chi_A^3 D_i\Phi - DD_i\Phi) \qquad (3.16)
\end{aligned}
$$

Substituting (2.20) and (3.15) gives

$$
\begin{aligned}
[D_i,\chi_A^1 + 2\theta\,\tau]\,\Phi &= \nabla_i\Phi - i[\Phi,(\chi_A^1+2\theta\,\tau)A_i] - \theta\epsilon_{jk}[\nabla_j\Phi,\nabla_k A_i]_+ \\
[D_i,\chi_A^2 - 2\theta\,\tau]\,\Phi &= -i[\Phi,(\chi_A^2-2\theta\,\tau)A_i] + \theta\epsilon_{jk}[\nabla_j\Phi,\nabla_k A_i]_+ \\
[D_i,\chi_A^3]\,\Phi &= \nabla_i\Phi - i[\Phi,\chi_A^3 A_i]\,, \qquad (3.17)
\end{aligned}
$$

where we used (2.32). The left hand sides of (3.17) can be computed directly. For this note that $D_i\Phi$ is covariant, and so the action of χ_A^a can be simply read off the results (3.7), (3.12) and (3.13)

$$
\begin{aligned}
(\chi_A^1 + 2\theta\,\tau)D_i\Phi &= \frac{\theta}{2}\epsilon_{jk}[A_j, D_kD_i\Phi + \nabla_k D_i\Phi]_+ + H^{(1)}_{D_i\Phi} \\
(\chi_A^2 - 2\theta\,\tau)D_i\Phi &= -\frac{\theta}{2}\epsilon_{jk}[A_j, D_kD_i\Phi + \nabla_k D_i\Phi]_+ + H^{(2)}_{D_i\Phi} \\
\chi_A^3 D_i\Phi &= H^{(3)}_{D_i\Phi}\,, \qquad (3.18)
\end{aligned}
$$

where the homogenous terms $H^{(a)}_{D_i\Phi}$ satisfy $H^{(3)}_{D_i\Phi} = H^{(1)}_{D_i\Phi} + H^{(2)}_{D_i\Phi}$. Using this result along with (3.7), (3.12) and (3.13) gives

$$
\begin{aligned}
[D_i,\chi_A^1 + 2\theta\tau]\Phi &= \theta\epsilon_{jk}\Big([F_{ij},D_k\Phi]_+ + [\nabla_j A_i,\nabla_k\Phi]_+ - \frac{i}{2}[\Phi,[A_k,F_{ij}-\nabla_j A_i]_+]\Big) \\
&\quad + D_iH^{(1)}_\Phi - H^{(1)}_{D_i\Phi} \\
[D_i,\chi_A^2 - 2\theta\tau]\Phi &= -\theta\epsilon_{jk}\Big([F_{ij},D_k\Phi]_+ + [\nabla_j A_i,\nabla_k\Phi]_+ - \frac{i}{2}[\Phi,[A_k,F_{ij}-\nabla_j A_i]_+]\Big) \\
&\quad + D_iH^{(2)}_\Phi - H^{(2)}_{D_i\Phi} \\
[D_i,\chi_A^3]\Phi &= D_iH^{(3)}_\Phi - H^{(3)}_{D_i\Phi} \qquad (3.19)
\end{aligned}
$$

Finally by comparing (3.17) with (3.19) we obtain the following general solution to $\chi_A^a A_i$

$$
\begin{aligned}
\chi_A^1 A_i &= -A_i + \frac{\theta}{2}\epsilon_{jk}[A_k,F_{ij}-\nabla_j A_i]_+ - 2\theta\tau(A_i) + H^{(1)}_{A_i} \\
\chi_A^2 A_i &= -\frac{\theta}{2}\epsilon_{jk}[A_k,F_{ij}-\nabla_j A_i]_+ + 2\theta\tau(A_i) + H^{(2)}_{A_i} \\
\chi_A^3 A_i &= -A_i + H^{(3)}_{A_i}\,, \qquad (3.20)
\end{aligned}
$$

where $H^{(a)}_{A_i}$ are homogeneous terms, i.e. they are covariant under gauge transformations. They are in general undetermined, apart from the following relations:

$$
\begin{aligned}
H^{(1)}_{D_i\Phi} &= D_i H^{(1)}_{\Phi} + \theta\epsilon_{jk}[F_{ij}, D_k\Phi]_+ + i[\Phi, H^{(1)}_{A_i}] - D_i\Phi \\
H^{(2)}_{D_i\Phi} &= D_i H^{(2)}_{\Phi} - \theta\epsilon_{jk}[F_{ij}, D_k\Phi]_+ + i[\Phi, H^{(2)}_{A_i}] \\
H^{(3)}_{D_i\Phi} &= D_i H^{(3)}_{\Phi} \qquad\qquad\qquad\qquad + i[\Phi, H^{(3)}_{A_i}] - D_i\Phi \ ,
\end{aligned}
\tag{3.21}
$$

in addition to $H^{(3)}_{A_i} = H^{(1)}_{A_i} + H^{(2)}_{A_i}$. If $H^{(3)}_{A_i}$ vanishes, then A_i is an eigenvector of χ^3_A with eigenvalue -1. (Minus the eigenvalue of χ^3_A is analogous to the conformal weight in conformal field theory.)

Variations δ^{s_a} of the field strength

$$
\begin{aligned}
\texttt{i)}\quad \delta^{s_1}_\epsilon F_{ij} = D_i\delta^{s_1}_\epsilon A_j - D_j\delta^{s_1}_\epsilon A_i &= \epsilon\left(\chi^1_A F_{ij} - \frac{1}{2}[\mathbf{x}_k, \nabla_k F_{ij}]_+ + 2\theta\,\tau(F_{ij})\right) \\
\texttt{ii)}\quad \delta^{s_2}_\epsilon F_{ij} = D_i\delta^{s_2}_\epsilon A_j - D_j\delta^{s_2}_\epsilon A_i &= \epsilon\,(\chi^2_A F_{ij} - 2\theta\nabla_\theta F_{ij}) \\
\texttt{iii)}\quad \delta^{s_3}_\epsilon F_{ij} = D_i\delta^{s_3}_\epsilon A_j - D_j\delta^{s_3}_\epsilon A_i &= \epsilon\,(\chi^3_A F_{ij} - DF_{ij})
\end{aligned}
\tag{3.22}
$$

are straightforward since F_{ij} is covariant. The action of χ^a_A can again be read off (3.7), (3.12) and (3.13)

$$
\begin{aligned}
(\chi^1_A + 2\theta\,\tau)F_{ij} &= \frac{\theta}{2}\epsilon_{jk}[A_j, D_kF_{ij} + \nabla_k F_{ij}]_+ + H^{(1)}_{F_{ij}} \\
(\chi^2_A - 2\theta\,\tau)F_{ij} &= -\frac{\theta}{2}\epsilon_{jk}[A_j, D_kF_{ij} + \nabla_k F_{ij}]_+ + H^{(2)}_{F_{ij}} \\
\chi^3_A F_{ij} &= H^{(3)}_{F_{ij}}
\end{aligned}
\tag{3.23}
$$

Using (3.22) some work shows that the homogeneous terms $H^{(a)}_{F_{ij}}$ are related to $H^{(a)}_{A_i}$ by

$$
\begin{aligned}
H^{(1)}_{F_{ij}} &= -2F_{ij} + D_i H^{(1)}_{A_j} - D_j H^{(1)}_{A_i} - \theta\epsilon_{k\ell}[F_{ik}, F_{j\ell}]_+ \\
H^{(2)}_{F_{ij}} &= D_i H^{(2)}_{A_j} - D_j H^{(2)}_{A_i} + \theta\epsilon_{k\ell}[F_{ik}, F_{j\ell}]_+ \\
H^{(3)}_{F_{ij}} &= -2F_{ij} + D_i H^{(3)}_{A_j} - D_j H^{(3)}_{A_i} \ ,
\end{aligned}
\tag{3.24}
$$

and thus satisfy $H^{(3)}_{F_{ij}} = H^{(1)}_{F_{ij}} + H^{(2)}_{F_{ij}}$. If $H^{(3)}_{A_i}$ vanishes, then F_{ij} is an eigenvector of χ^3_A with eigenvalue -2. For Seiberg-Witten transformations $H^{(2)}_{A_i}$ is commonly set to zero, implying $H^{(2)}_{F_{ij}} = \theta\epsilon_{k\ell}[F_{ik}, F_{j\ell}]_+$.

3.3 One parameter family of Seiberg-Witten transformations

In section 2.2, we considered a one parameter family of Kontsevich maps (2.36) which led to a one-parameter family of star products (2.38) connecting the Voros and Moyal Weyl products. In that case $\tau_\beta = \frac{\beta}{4}\,\nabla_i\nabla_i$, and we have a one parameter family of Seiberg-Witten variations on the gauge fields and gauge parameter

$$\begin{aligned}
\delta^{SW}\Phi &= \delta\theta\Big(-\frac{1}{4}\epsilon_{ij}[A_i,(D_j+\nabla_j)\Phi]_+ + \frac{\beta}{4}\,\nabla_i\nabla_i\Phi + \frac{1}{2\theta}H^{(2)}_\Phi\Big)\\
\delta^{SW}A_i &= \delta\theta\Big(-\frac{1}{4}\epsilon_{jk}[A_k,F_{ij}-\nabla_jA_i]_+ + \frac{\beta}{4}\,\nabla_j\nabla_jA_i + \frac{1}{2\theta}H^{(2)}_{A_i}\Big)\\
\delta^{SW}F_{ij} &= \delta\theta\Big(-\frac{1}{4}\epsilon_{jk}[A_j,(D_k+\nabla_k)F_{ij}]_+ + \frac{\beta}{4}\,\nabla_k\nabla_kF_{ij} + \frac{1}{2\theta}H^{(2)}_{F_{ij}}\Big)\ ,\\
\delta^{SW}\Lambda &= \delta\theta\Big(\frac{1}{4}\epsilon_{ij}[A_i,\nabla_j\Lambda]_+ + \frac{\beta}{4}\,\nabla_i\nabla_i\Lambda\Big) + \Lambda^{(2)}_\epsilon
\end{aligned} \tag{3.25}$$

Using the identities

$$\begin{aligned}
D_iD_i\Phi &= \nabla_i\nabla_i\Phi - i[A_i,(D_i+\nabla_i)\Phi] + i[\Phi,\nabla_iA_i]\\
D_jF_{ji} &= \nabla_j\nabla_jA_i + i[A_j,F_{ij}-\nabla_jA_i] - D_i\nabla_jA_j\ ,
\end{aligned} \tag{3.26}$$

the transformations of the gauge fields in (3.25) can be re-expressed, up to gauge transformations, as

$$\begin{aligned}
\delta^{SW}\Phi &= \delta\theta\Big(-\frac{1}{4}\epsilon_{ij}[A_i,(D_j+\nabla_j)\Phi]_+ + \frac{i\beta}{4}\,[A_i,(D_i+\nabla_i)\Phi] + \frac{1}{2\theta}\tilde H^{(2)}_\Phi\Big)\\
\delta^{SW}A_i &= \delta\theta\Big(-\frac{1}{4}\epsilon_{jk}[A_k,F_{ij}-\nabla_jA_i]_+ - \frac{i\beta}{4}\,[A_j,F_{ij}-\nabla_jA_i] + \frac{1}{2\theta}\tilde H^{(2)}_{A_i}\Big)\\
\delta^{SW}F_{ij} &= \delta\theta\Big(-\frac{1}{4}\epsilon_{jk}[A_j,(D_k+\nabla_k)F_{ij}]_+ + \frac{i\beta}{4}\,[A_k,(D_k+\nabla_k)F_{ij}] + \frac{1}{2\theta}\tilde H^{(2)}_{F_{ij}}\Big)\ ,
\end{aligned} \tag{3.27}$$

where we redefined the homogeneous terms:

$$\begin{aligned}
\tilde H^{(2)}_\Phi &= H^{(2)}_\Phi + \frac{\beta\theta}{2}\,D_iD_i\Phi\\
\tilde H^{(2)}_{A_i} &= H^{(2)}_{A_i} + \frac{\beta\theta}{2}D_jF_{ji}\\
\tilde H^{(2)}_{F_{ij}} &= H^{(2)}_{F_{ij}} + \frac{\beta\theta}{2}\,D_kD_kF_{ij}
\end{aligned} \tag{3.28}$$

For $\beta = 1$ we get the Seiberg-Witten map for the Voros star product, while for $\beta = 0$ it is the Seiberg-Witten map for the Moyal Weyl star product. The $\beta-$dependent inhomogeneous terms in (3.27) are expressed in terms of commutators, which then vanish at lowest order in θ. From (3.27), τ_β acting on fields belonging to nontrivial representations of the gauge group has the correct $\theta \to 0$ limit, which was not obvious from (3.25). In going from (3.25) to (3.27)

we have absorbed the lowest order θ contributions to the $\beta-$dependent inhomogeneous terms of (3.25) into the homogeneous terms. The $\theta \to 0$ limit is then in agreement with (1.4), and consistent with the fact the first order result for the Seiberg-Witten map should not depend on the choice of star product.

3.4 Covariant position operator

For gauge theories on the noncommutative plane one can define a covariant position operator X_i:

$$X_i = \frac{\hat{\mathtt{x}}_i}{\theta} + \epsilon_{ij} A_j \ , \tag{3.29}$$

where $\hat{\mathtt{x}}_i$ are gauge invariant functions such that $\hat{\mathtt{x}}_i(\mathtt{x}, \theta) = \mathtt{x}_i$. One can obtain the variations $\delta^{s_a}_\epsilon X_i$ by computing the action of χ^a_A. The Seiberg-Witten variations $\delta^{SW} X_i$ were previously obtained in [12] for the Moyal Weyl case $\tau = 0$. Since X_i is covariant, the action of χ^a_A on it can again be read off (3.7), (3.12) and (3.13)

$$\begin{aligned}
(\chi^1_A + 2\theta\,\tau) X_i &= -2\epsilon_{ij} A_j + \frac{\theta}{2}\Big([A_j, F_{ji} - \nabla_i A_j]_+ + [A_i, \nabla_j A_j]_+ \Big) + H^{(1)}_{X_i} \\
(\chi^2_A - 2\theta\,\tau) X_i &= 2\epsilon_{ij} A_j - \frac{\theta}{2}\Big([A_j, F_{ji} - \nabla_i A_j]_+ + [A_i, \nabla_j A_j]_+ \Big) + H^{(2)}_{X_i} \\
\chi^3_A X_i &= H^{(3)}_{X_i}
\end{aligned} \tag{3.30}$$

where $H^{(a)}_{X_i}$ are homogeneous terms, satisfying $H^{(3)}_{X_i} = H^{(1)}_{X_i} + H^{(2)}_{X_i}$. Using (3.20), we can compare this with χ^a_A acting on $\epsilon_{ij} A_j$, and then deduce the action of χ^a_A on the ratio of the noncommutative coordinate with θ. Up to homogeneous terms, one gets

$$\begin{aligned}
(\chi^1_A + 2\theta\,\tau)\, \frac{\hat{\mathtt{x}}_i}{\theta} &= \frac{\hat{\mathtt{x}}_i}{\theta} \\
(\chi^2_A - 2\theta\,\tau)\, \frac{\hat{\mathtt{x}}_i}{\theta} &= -2\, \frac{\hat{\mathtt{x}}_i}{\theta} \\
\chi^3_A\, \frac{\hat{\mathtt{x}}_i}{\theta} &= -\ \frac{\hat{\mathtt{x}}_i}{\theta}
\end{aligned} \tag{3.31}$$

Because variations $\delta^{s_a}_\epsilon$ are evaluated at fixed coordinates on the noncommutative plane and fixed values of θ, the variations of $\hat{\mathtt{x}}_i$ and θ, and consequently their ratio, should vanish. This follows if (3.31) is an exact result, *including the homogeneous terms*. As a result the homogeneous terms $H^{(a)}_{X_i}$ and $H^{(a)}_{A_i}$ are related by

$$\begin{aligned}
H^{(1)}_{X_i} &= \epsilon_{ij} H^{(1)}_{A_j} + \ X_i \\
H^{(2)}_{X_i} &= \epsilon_{ij} H^{(2)}_{A_j} - 2X_i \\
H^{(3)}_{X_i} &= \epsilon_{ij} H^{(3)}_{A_j} - \ \ X_i
\end{aligned} \tag{3.32}$$

3.5 Constraints on the Homogeneous Terms

In the above we have found a number of relations connecting the various homogeneous terms: (3.21) and (3.24). From (3.21) it follows that not all terms $H^{(a)}_{D_i\Phi}$, $H^{(a)}_{\Phi}$ and $H^{(a)}_{A_i}$, for any a, can simultaneously vanish. From (3.24), $H^{(a)}_{A_i}$ and $H^{(a)}_{F_{ij}}$, for any a, cannot simultaneously vanish. We can get additional constraints on the homogenous terms if we demand that the gauge fields carry a faithful representation of the two independent scale transformations `i)` and `ii)`. When including this demand there appear to be no obstructions to having nonvanishing homogeneous terms. We give some explicit solutions. The new conditions are however insufficient in removing all the ambiguities in the homogeneous terms. More constraints may result from the presence of other symmetries (although they won't be considered here) and they may help fix further degrees of freedom in the homogeneous terms.

At first order, the two independent scale transformations `i)` and `ii)` transformations commute. An analogous statement can be made at all orders using the scale generators. Acting on an arbitrary function F, they are $\frac{1}{2}[\mathtt{x}_i, \nabla_i F]_+$ and $2\theta(\nabla_\theta - \tau)F$. Their commutator acting on F vanishes,

$$\begin{aligned} & \frac{1}{2}[\mathtt{x}_i, \nabla_i\, 2\theta(\nabla_\theta - \tau)F]_+ - 2\theta(\nabla_\theta - \tau)\ \frac{1}{2}[\mathtt{x}_i, \nabla_i F]_+ \\ = &\ \frac{1}{2}\Big([\mathtt{x}_i, \nabla_i\ DF]_+ - D[\mathtt{x}_i, \nabla_i F]_+ \Big) \\ = &\ \frac{1}{2}\Big([\mathtt{x}_i, [\nabla_i, D]F]_+ - [D\mathtt{x}_i, \nabla_i F]_+ \Big) \ = \ 0\ , \end{aligned} \tag{3.33}$$

where we used (2.18). So for gauge fields to carry a faithful representation we need that `i)` and `ii)` commute on the space of such fields. (More generally, one only needs that the commutator of `i)` and `ii)` is a gauge transformation.)

We now compute $[\delta^{s_1}_{\epsilon_1}, \delta^{s_2}_{\epsilon_2}]$. Acting on Φ we get

$$\begin{aligned} [\delta^{s_1}_{\epsilon_1}, \delta^{s_2}_{\epsilon_2}]\ \Phi \ &= \ [\delta^{s_3}_{\epsilon_1}, \delta^{s_2}_{\epsilon_2}]\ \Phi = \epsilon_1\epsilon_2 \Big\{ \chi^3_A H^{(2)}_\Phi - (\chi^2_A - 2\theta\ \tau) H^{(3)}_\Phi \\ &- \ \frac{\theta}{2}\epsilon_{ij}\ \Big([A_i, (D_j + \nabla_j) H^{(3)}_\Phi + i[\Phi, H^{(3)}_{A_j}]\,]_+ + [H^{(3)}_{A_i}, (D_j + \nabla_j)\Phi]_+ \Big) \Big\} \end{aligned} \tag{3.34}$$

An obvious solution to $[\delta^{s_1}_{\epsilon_1}, \delta^{s_2}_{\epsilon_2}]\ \Phi = 0$ is $H^{(3)}_{A_i} = H^{(a)}_\Phi = 0\ ,\ \ a = 1, 2, 3$. More general solutions are also possible. For example, setting $H^{(3)}_{A_i} = 0$, and $H^{(2)}_\Phi$ and $H^{(3)}_\Phi$ equal to functions only of Φ we need that

$$H^{(2)}_\Phi\ \frac{\delta}{\delta\Phi}\ H^{(3)}_\Phi = H^{(3)}_\Phi\ \frac{\delta}{\delta\Phi}\ H^{(2)}_\Phi\ , \tag{3.35}$$

using (3.7) and (3.13). (3.35) is then solved for $H^{(3)}_\Phi$, $H^{(2)}_\Phi$, and consequently $H^{(1)}_\Phi$, proportional to the same function of Φ.

Acting on A_i the commutator gives

$$
\begin{aligned}
[\delta^{s_1}_{\epsilon_1}, \delta^{s_2}_{\epsilon_2}]\, A_i &= [\delta^{s_3}_{\epsilon_1}, \delta^{s_2}_{\epsilon_2}]\, A_i \\
&= \epsilon_1\epsilon_2 \Big\{ (\chi^3_A + 1) H^{(2)}_{A_i} - (\chi^2_A \;-\; 2\theta\,\tau) H^{(3)}_{A_i} \\
&\quad -\frac{\theta}{2}\epsilon_{jk}\Big([A_k, D_i H^{(3)}_{A_j} - D_j H^{(3)}_{A_j} - \nabla_j H^{(3)}_{A_i}]_+ + [H^{(3)}_{A_k}, F_{ij} - \nabla_j A_i]_+ \Big) \Big\} \;,
\end{aligned}
\tag{3.36}
$$

where we used (3.24). $[\delta^{s_1}_{\epsilon_1}, \delta^{s_2}_{\epsilon_2}]\, A_i = 0$ is obviously satisfied for $H^{(a)}_{A_i} = 0\;,\;\; a = 1, 2, 3\;$. It is not difficult to construct other solutions to $[\delta^{s_1}_{\epsilon_1}, \delta^{s_2}_{\epsilon_2}]\, A_i = 0$. For example, if there is a covariant vector V_i which is an eigenvector of χ^3_A with eigenvalue -1, we can satisfy (3.36) by setting

$$
H^{(1)}_{A_i} = -H^{(2)}_{A_i} = V_i \;, \qquad H^{(3)}_{A_i} = 0 \tag{3.37}
$$

For the case of pure gauge theory, we can let V_i be proportional to the covariant position operator X_i. It has eigenvalue -1, since from (3.32), $H^{(3)}_{X_i} = -X_i$. When other fields are present, V_i can have additional contributions. For example in scalar field theory such a contribution can be chosen to be proportional to $D_i\Phi$ provided that $H^{(3)}_{\Phi} = 0$, and using (3.19).

4 Concluding Remarks

We have seen that one of the previous transformations, namely `iii)`, preserves the fundamental commutation relations. It was generated by D which satisfies the Leibniz rule, as well as annihilates the left hand side of (1.1), resulting in a symmetry of the algebra. Here we remark on the possibility of implementing transformation `iii)` as a symmetry of the dynamics as well. We note below that this symmetry is not a deformation of the standard dilation symmetry on the (commutative) plane, but rather a new symmetry on the noncommutative plane $\times\ \mathbb{R}$. For simplicity, we shall restrict the remarks here to lowest order in θ. We plan to give a more thorough discussion on this topic in a later article.

Since `iii)` involves re-scalings in both x_i and θ, we cannot keep θ fixed. So we should not consider a single noncommutative plane, but rather an ensemble of such noncommutative planes. θ characterizing the noncommutative plane is often related to some external field. (In string theory it is associated with the external 2-form on the brane, while in the Hall effect it is the external magnetic field.) From that point of view we are then allowing for changing values of the field. Let us assume the changes occur in some (commuting) time variable t, i.e. the field and therefore θ are functions of the time t. From a cosmological perspective, it is tempting to imagine a $\theta(t)$ where θ monotamicaly decreases with increasing t, for then one would have a model where noncommutativity was significant at early times, while it is negligible for late (or present) times.

At first order we write the action S as an integral of the Lagrangian density $L(x, t)$ over

$\mathbb{R}^3$, here parametrized by $(x_1,\ x_2, t)$

$$S = \int dt d^2x\ L(x,t) \tag{4.1}$$

If we assume that $\theta(t)$ is a nonsingular function, we can do a change of variables

$$S = \int d\theta d^2x\ \mathcal{L}(x,\theta)\ , \qquad L(x,t) = \left|\frac{d\theta}{dt}\right| \mathcal{L}(x,\theta) \tag{4.2}$$

This action is invariant under `iii`) when the Lagrangian density $\mathcal{L}(x,\theta)$ transforms as

$$\mathcal{L}(x,\theta) \rightarrow {}^{\rho_3}\mathcal{L}(x,\theta) = \rho^4\ \mathcal{L}(\rho x, \rho^2\theta) \tag{4.3}$$

The corresponding infinitesimal transformation is

$$\mathcal{L}(x,\theta) \rightarrow \mathcal{L}(x,\theta) + \delta^{s_3}_\epsilon \mathcal{L}(x,\theta)\ , \tag{4.4}$$

where variation of the Lagrangian density is a total divergence in $\mathbb{R}^3$,

$$\delta^{s_3}_\epsilon \mathcal{L}(x,\theta)\ =\ -\epsilon\ (4+D)\mathcal{L}(x,\theta)\ =\ \partial_i C^i + \partial_\theta C^\theta\ ,$$

$$C^i = -\epsilon x_i \mathcal{L} \qquad C^\theta = -2\epsilon\theta\mathcal{L} \tag{4.5}$$

In terms of the previous notation $\chi^3_{\mathcal{A}}\mathcal{L} = -4\ \mathcal{L}$, or $\mathcal{L}$ has conformal weight four. It is easy to write a Lagrangian field theory with this property, as Lagrangian densities associated with *four* dimensional conformal symmetry transform in the same manner under dilations. An example is scalar field theory with a ϕ^4 interaction. At zeroth order in θ, the Lagrangian density is

$$\mathcal{L} = \frac{1}{2}(\partial_i\phi)^2 + \frac{g}{4!}\phi^4\ , \tag{4.6}$$

where g is a dimensionless coupling and the scalar field $\phi(x,\theta)$ satisfies $\chi^3_{\mathcal{A}}\phi = -\ \phi$, or

$$\delta^{s_3}_\epsilon \phi(x,\theta) = -\epsilon\ (1+D)\phi(x,\theta) \tag{4.7}$$

Thus the conformal weight of $\phi(x,\theta)$ is one, in agreement with scalar fields in four dimensional conformal field theory. As in four dimensions, the mass term breaks the scale symmetry.

From Noether's theorem the above symmetry implies a conserved current, but it is a conserved current in $\mathbb{R}^3$. The current is not conserved on two dimensional θ-slices of $\mathbb{R}^3$. So denoting the current as j^μ, $\mu = 1, 2, \theta$, the conservation law is

$$\partial_i j^i + \partial_\theta j^\theta = 0 \tag{4.8}$$

For generic fields ψ_α the current is given by

$$\epsilon j^\mu = \frac{\partial\mathcal{L}}{\partial\partial_\mu\psi_\alpha}\delta^{s_3}_\epsilon\psi_\alpha - C^\mu \tag{4.9}$$

In the example of the scalar field ϕ with Lagrangian density (4.6) we then get

$$j^i = -\partial_i\phi(1+D)\phi + \frac{1}{2}x_i(\partial_j\phi)^2 + \frac{g}{4!}x_i\phi^4 \qquad\qquad j^\theta = 2\theta\left(\frac{1}{2}(\partial_j\phi)^2 + \frac{g}{4!}\phi^4\right) \tag{4.10}$$

We note that although j^θ vanishes in the commutative limit $\theta \to 0$, it nevertheless gives a nonzero contribution to the divergence when $\theta \to 0$, and so the current is not conserved when restricted to the $\theta = 0$ slice of $\mathbb{R}^3$. For the example of the scalar field, $\partial_i j^i|_{\theta=0} = -2\,\mathcal{L}$. (More generally, this holds provided $\mathcal{L}$ doesn't depend on θ-derivatives of the fields.) The symmetry transformation then cannot be regarded as a deformation of the standard dilation symmetry on the plane. For the free scalar field, the latter is associated with variations $\delta^{s_1}_\epsilon\phi = -\epsilon\, x_i\partial_i\phi$, corresponding to conformal weight zero, which differs from the $\theta \to 0$ limit of variations (4.7), $\delta^{s_3}_\epsilon\phi|_{\theta=0} = -\epsilon\,(1 + x_i\partial_i)\phi$, having conformal weight one.

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REFERENCES

[1] N. Seiberg and E. Witten, JHEP **9909**, 032 (1999).

[2] P. A. Horvathy, L. Martina and P. C. Stichel, Phys. Lett. B **564**, 149 (2003); P. A. Horvathy and M. S. Plyushchay, arXiv:hep-th/0404137.

[3] T. Asakawa and I. Kishimoto, JHEP **9911**, 024 (1999).

[4] B. Jurco, L. Moller, S. Schraml, P. Schupp and J. Wess, Eur. Phys. J. C **21**, 383 (2001).

[5] D. Brace, B. L. Cerchiai, A. F. Pasqua, U. Varadarajan and B. Zumino, JHEP **0106**, 047 (2001); D. Brace, B. L. Cerchiai and B. Zumino, Int. J. Mod. Phys. A **17S1**, 205 (2002).

[6] G. Barnich, F. Brandt and M. Grigoriev, JHEP **0208**, 023 (2002).

[7] J. M. Grimstrup, T. Jonsson and L. Thorlacius, JHEP **0312**, 001 (2003); J. M. Grimstrup, Acta Phys. Polon. B **34**, 4855 (2003).

[8] H. Groenewold, Physica (Amsterdam) **12**, 405 (1946); J. Moyal, Proc. Camb. Phil. Soc. **45**, 99 (1949).

[9] F. Bayen, in *Group Theoretical Methods in Physics*, ed. E. Beiglböck , et. al. [Lect. Notes Phys. **94**, 260 (1979)]; A. Voros, Phys. Rev. **A 40**, 6814 (1989).

[10] C. K. Zachos, J. Math. Phys. **41**, 5129 (2000).

[11] G. Alexanian, A. Pinzul and A. Stern, Nucl. Phys. B **600**, 531 (2001).

[12] A. P. Polychronakos, Annals Phys. **301**, 174 (2002); R. Jackiw, S. Y. Pi and A. P. Polychronakos, Annals Phys. **301**, 157 (2002).